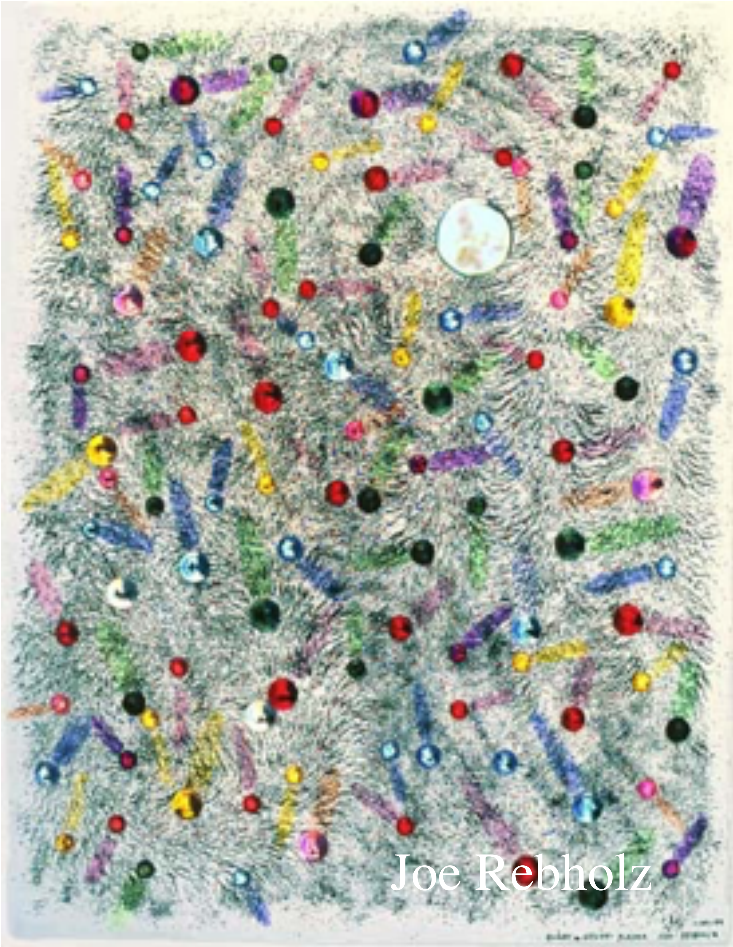
On Wave Turbulence in Non-Abelian Plasmas

(In preparation)

Yacine Mehtar-Tani
INT, University of Washington

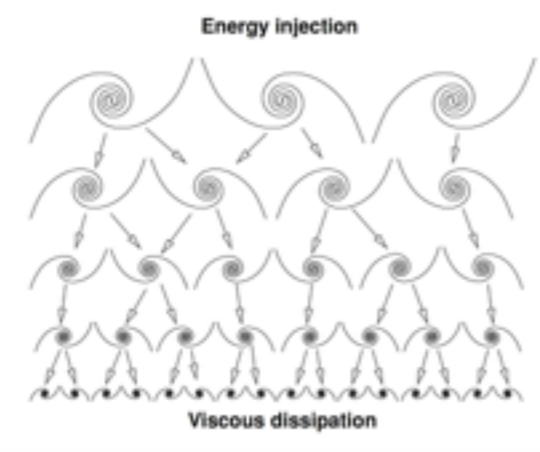


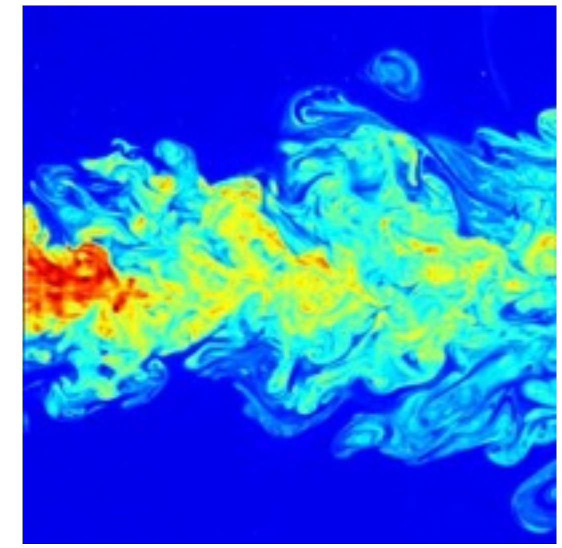
New Progress in Heavy Ion Collisions:
What is hot in the QGP?

October 06, 2015 Wuhan, China

Wave Turbulence (I)

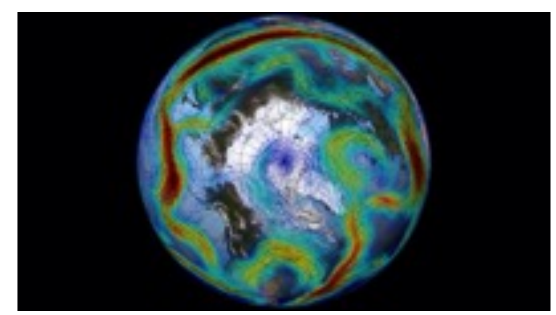
- Out-of-equilibrium statistics of random non-linear waves

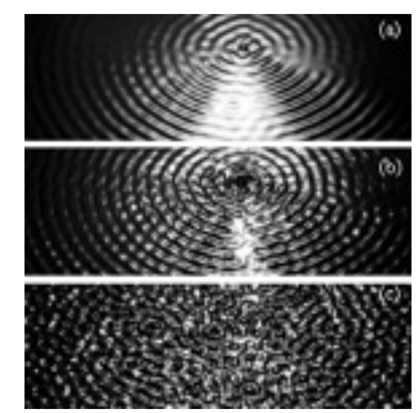


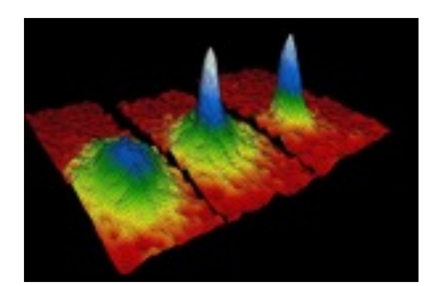


- Similarity with fluid turbulence: inviscid transport of conserved quantities from large to small scales through the so-called transparency window (or inertial range)

- Some examples:
  - Atmospheric Rossby waves
  - Water surface gravity and capillary waves
  - Waves in plasmas
  - Nonlinear Schrödinger equation (NL Optics, BEC)







Wave Turbulence (II)

- Waves are excited by external processes. Driven turbulence: Open system with source and sink → away from thermodynamical equilibrium

- Steady states characterized by constant fluxes P and Q rather than temperature and thermodynamical potentials

- Kolmogorov-Obukhov (KO41) theory relies on Locality of interactions: Only eddies (waves) with comparable sizes (wavelengths) interact. Steady state power spectra in momentum space depend on the fluxes and not on the pumping and dissipation scales

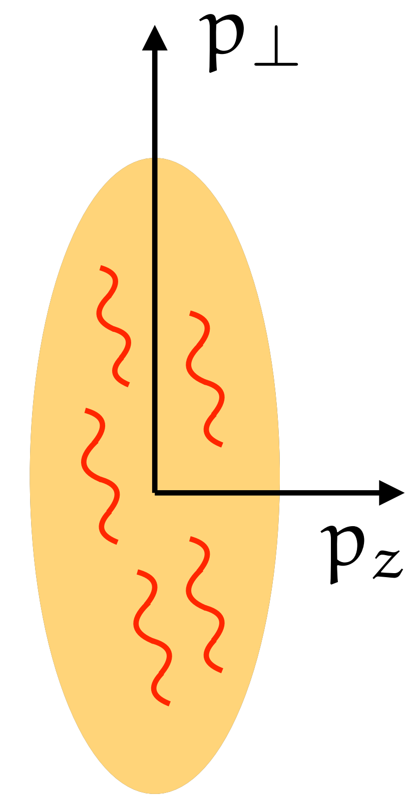
- Weak (Wave) Turbulence Theory: Kinetic description in the limit of weak nonlinearities (NB: no theory for strong turbulence)

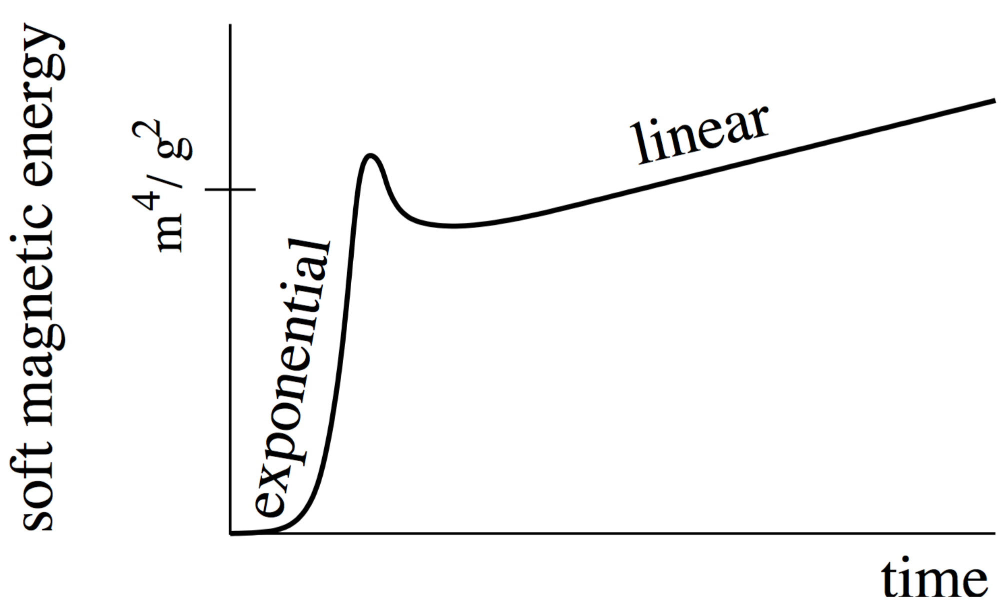
V. E. Zakharov, V. S. L'vov, G. Falkovich (Springer- Verlag, 1992)

Turbulence in early stages of Heavy Ion Collisions

- Immediately after the collision the system is far from equilibrium. Anisotropic particle distribution in momentum space.

- Chromo-Weibel Instabilities : Momentum anisotropy induces exponential growth of soft modes - transverse magnetic and electric fields - (early stage: as in abelian plasmas) which turns into a linear growth due to nonlinear interactions inherent to non-abelian plasmas

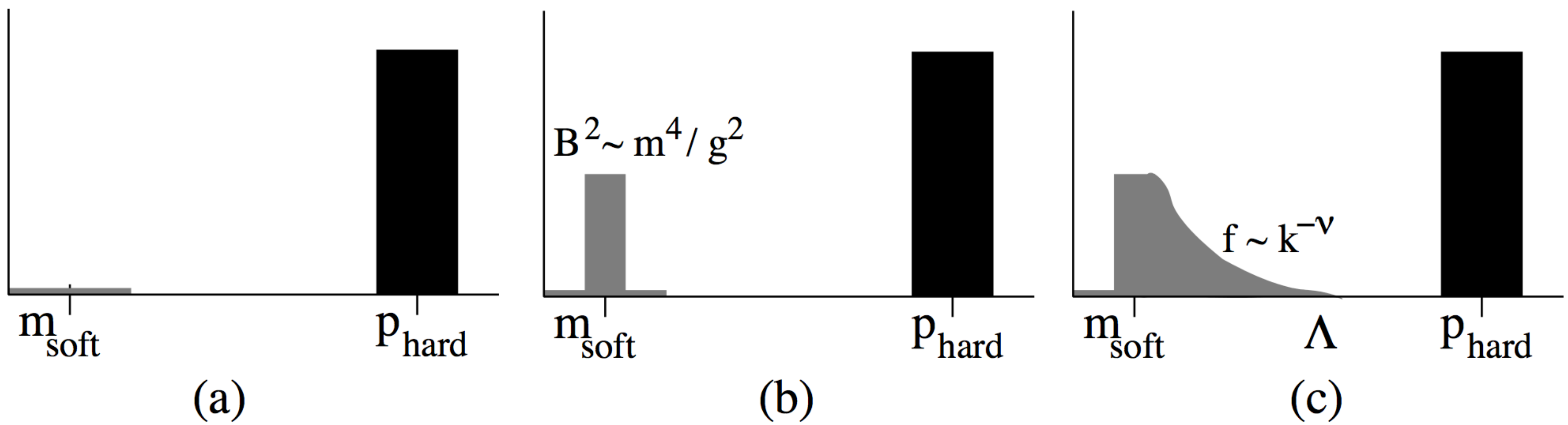




E. S. Weibel (1959)
S. Mrowczynski (1993)
P. Arnold, J. Lenaghan, G. Moore, L. Yaffe (2005)
A. Rebhan, P. Romatschke, and M. Strickland (2005)
D. Bödeker, K. Rummukainen (2005)
P. Arnold, G. D. Moore (2005)

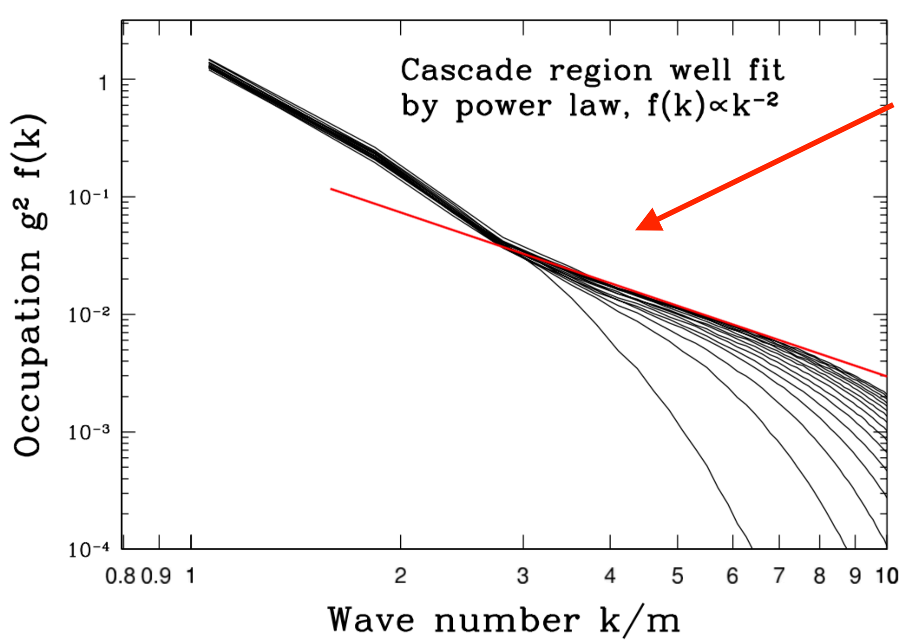
Turbulence in early stages of Heavy Ion Collisions

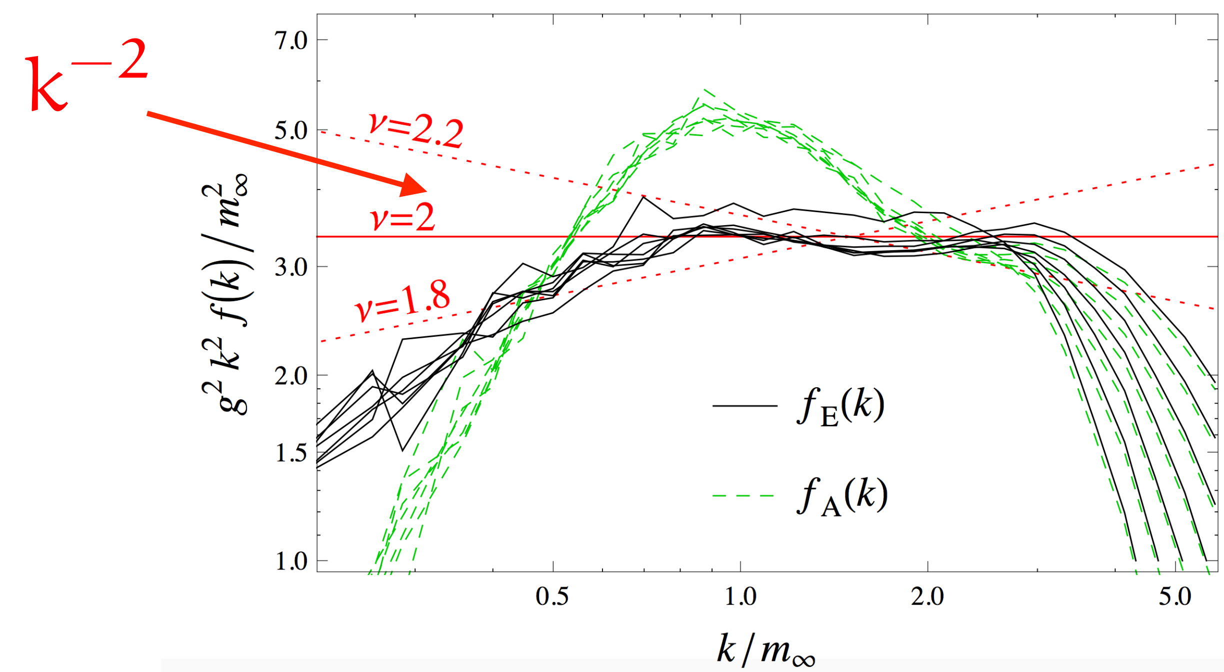
- Hard-loop simulations (large scale separation between hard modes and soft excitations) : Nonlinear interactions develop a turbulent cascade in the UV with exponent 2



P.Arnold, G. D. Moore (2005)

A. Ipp, A. Rebhan, M. Strickland (2011)





Weak Turbulence in Kinetic Theory

- Can one understand this power spectrum k^{-2} from first principles?

- Note: from A. H. Mueller, A. I. Shoshi, S. M. H. Wong (2006):

  Turbulence in QCD is nonlocal $\Rightarrow$ $n(k) \sim k^{-1}$

- Some caveats (in this work):

  - Homogeneous and isotropic system of gluons
  - Forcing: Energy injection with constant rate P at $k_f \gg m$: Dispersion relation $\omega(k) \equiv |k|$
  - Weak nonlinearities in the classical limit (high occupancy):

$$g^2 \ll 1 \qquad \text{and} \qquad 1 \ll n(k) \ll \frac{1}{g^2}$$

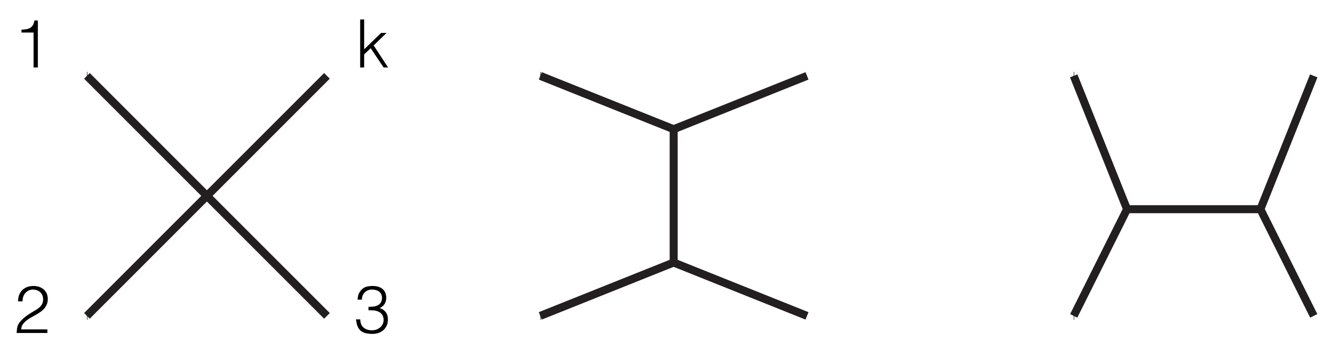
Elastic 2 to 2 process (4-waves interactions)

- Elastic gluon-gluon scattering

$$\frac{\partial}{\partial t} n_k = \frac{1}{2}\int_{k_1,k_2,k_3} \frac{1}{2\omega(k)} |\mathcal{M}_{12\to 3k}|^2\, \delta(\sum_i k_i)\delta(\sum_i \omega_i)\ F[n]$$

$$F[n] \equiv [n_{k_1} n_{k_2} n_k + n_{k_1} n_{k_2} n_{k_3} - n_{k_1} n_{k_3} n_k - n_{k_2} n_{k_3} n_k] \sim n^3$$

$$\mathcal{M}_{12\to 3k} \equiv$$



- Two constant of motion: particle number and energy $\Rightarrow$ Two fluxes

$$Q \equiv \dot{N} = \int d^3k\, \dot{n}(k) \qquad\qquad P \equiv \dot{E} = \int d^3k\, |k|\, \dot{n}(k)$$

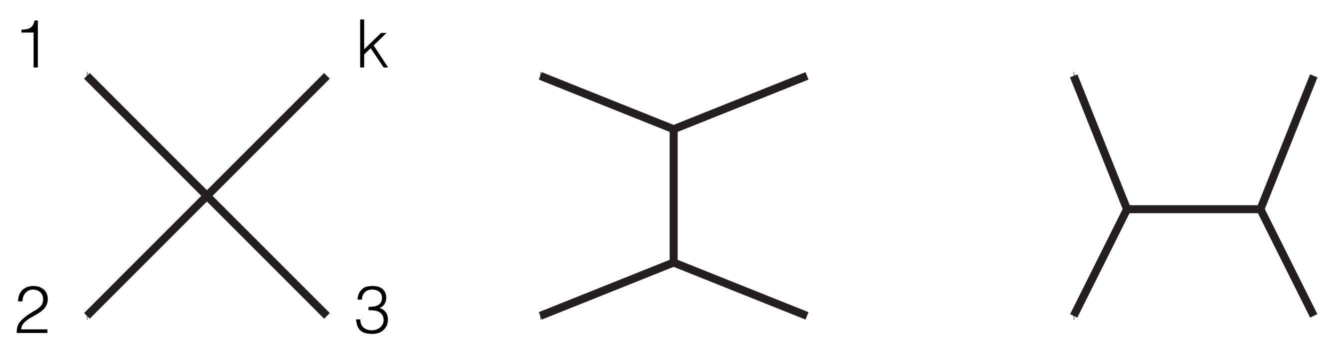
Elastic 2 to 2 process (4-waves interactions)

- Elastic gluon-gluon scattering

$$\frac{\partial}{\partial t} n_k = \frac{1}{2} \int_{k_1,k_2,k_3} \frac{1}{2\omega(k)} |\mathcal{M}_{12\to 3k}|^2 \, \delta(\sum_i k_i)\delta(\sum_i \omega_i) \; F[n]$$

$$F[n] \equiv [n_{k_1} n_{k_2} n_k + n_{k_1} n_{k_2} n_{k_3} - n_{k_1} n_{k_3} n_k - n_{k_2} n_{k_3} n_k] \sim n^3$$

$$\mathcal{M}_{12\to 3k} \equiv$$



- H-theorem $\Rightarrow$ Thermal fixed-point (vanishing fluxes) $P = Q = 0$

$$n_k = \frac{T}{\omega(k) - \mu}$$

(Rayleigh-Jeans distribution)

Kolmogorov-Zakharov (KZ) Spectra

- From collision integral the flus scales as the cube of the the occupation numebr: nonlinear 4-wave interactions

$$P \sim Q \sim \dot{n} \sim n^3 \quad \Rightarrow n \sim P^{1/3} \sim Q^{1/3}$$

- Dimensional analysis determines uniquely the out-of-equilibrium steady state (KZ) power spectra if the interactions are local in momentum space

$$n(k) \sim \frac{Q^{1/3}}{k^{4/3}} \qquad\qquad n(k) \sim \frac{P^{1/3}}{k^{5/3}}$$

particle cascade | energy cascade

- Same exponents for scalar theories in the absence of condensation

Dual cascade: Fjørthoft argument (1953)

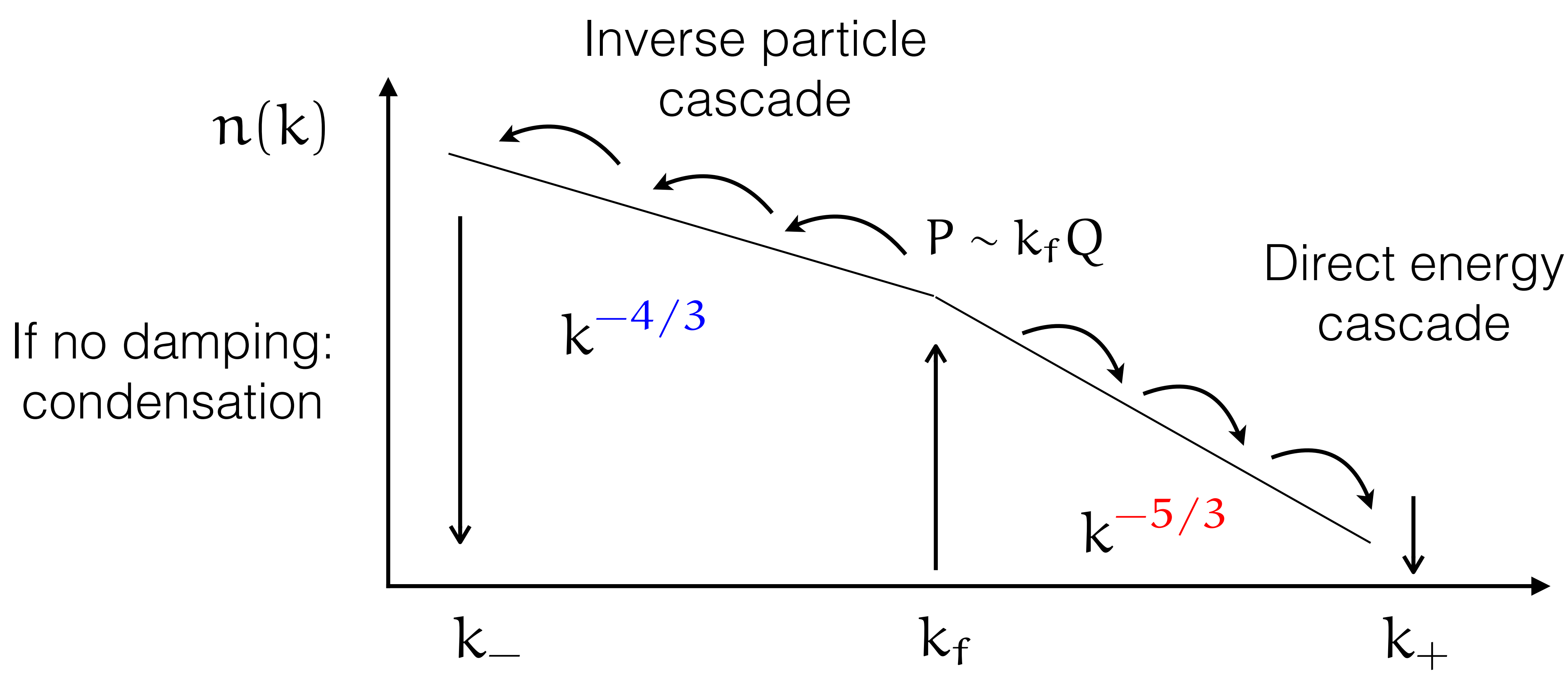
- Q: Direction of fluxes? Injection of energy at k_f and dissipating at

$$k_- \ll k_f \ll k_+$$

- *Reductio ad absurdum:* If energy was dissipating at low momenta then particles would dissipate faster than the pumping rate! $\Longrightarrow$ Direct energy cascade

$$Q_- \sim \frac{P}{k_-} \gg \frac{P}{k_f} \sim Q$$

absurd

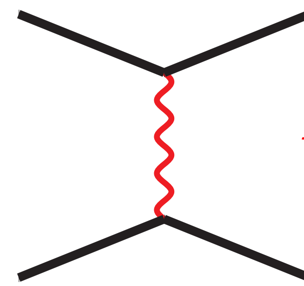


Are KZ spectra in QCD physically relevant?

Inelastic scattering in the small angle approximation

- Coulomb interaction is singular at small momentum transfer $k \gg q \geq m$

$$|\mathcal{M}_{k1\to 23}|^2 \sim \alpha^2 \frac{s^2}{t^2}$$



$t \sim q^2 \ll s$

- Small angle approx: Fokker-Plank equation: Diffusion and drag

$$\frac{\partial}{\partial t} n_k \equiv \frac{\hat{q}}{4k^2}\frac{\partial}{\partial k} k^2 \left[\frac{\partial}{\partial k} n_k + \frac{n_k^2}{T_*}\right]$$

L. D. Landau (1937) B. Svetitski (1988)

Diffusion coefficient

$$\hat{q} \equiv \sim \alpha^2 \int d^3k \, n_k^2$$

Screening mass

$$m^2 \sim \alpha \int \frac{d^3k}{|k|} n_k$$

Effective temperature

$$T_* \sim \frac{\hat{q}}{\alpha m^2}$$

- KZ spectra are not stationary solutions of the collision integral (contrary to non-relativistic Coulomb scattering! A. V. Kats, V. M. Kontorovich, S. S. Moiseev, and V. E. Novikov (1975))

- Furthermore: $\hat{q}$ diverges in the IR for $n \sim k^{-5/3}$ and for $n \sim k^{-4/3}$ in the UV

Steady state solutions

$$\frac{\partial}{\partial t} n_k \equiv \frac{\hat{q}}{4k^2} \frac{\partial}{\partial k} k^2 \left[\frac{\partial}{\partial k} n_k + \frac{n_k^2}{T_*} \right] + F - D$$

Forcing

Dumping

- Thermal fixed point: $\frac{T_*}{k-\mu}$
- Non-thermal fixed point (inverse particle cascade): $n(k) \sim \frac{A}{k} > \frac{T_*}{k}$

$$A \equiv \frac{1}{2} T_* \left(1 + \sqrt{1 + \frac{16Q}{\hat{q} T_*}} \right)$$

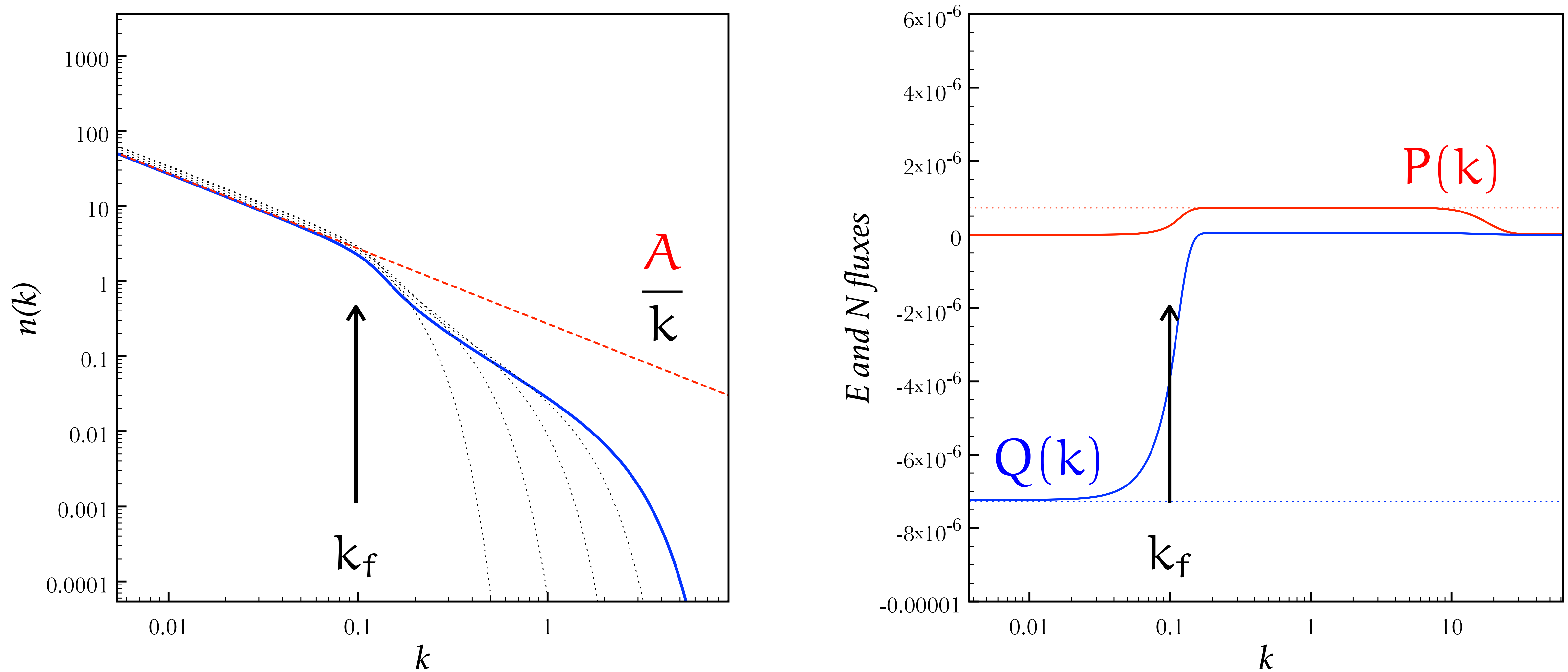
- Warm cascade behavior:

2-D Optical turbulence: S. Dyachenko, A.C. Newell, A. Pushkarev, V.E. Zakharov (1992)

Boltzmann equation: D. Proment, S. Nazarenko, P. Asinari, and M. Onorato (2011)

- No stationary solution for the energy cascade without a sink

Numerical simulation of FK equation with forcing



- The occupation number (left) and, the energy and particle number fluxes (right) at late times

- Constant particle flux at k=0 $\Rightarrow$ Bose-Einstein condensate

Contribution from inelastic processes?

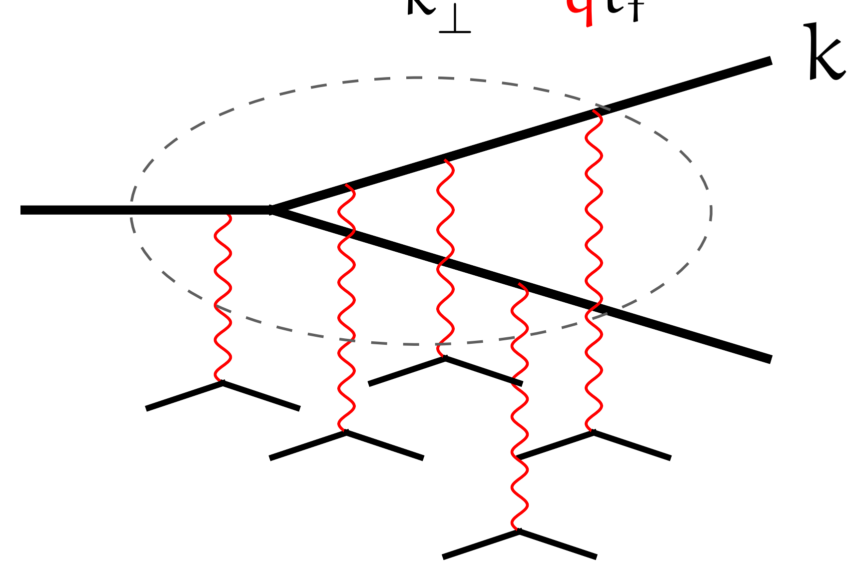
- Naively we would expect inelastic processes to be suppressed by powers of the coupling constant
- In non-abelian plasmas inelastic processes are enhanced due to collinear divergences and hence cannot be neglected compared to elastic processes
- In what follows we shall proceed in the small angle approximation: $2 \to 3$ process reduces to an effective $1 \to 2$

Effective 3 waves interaction (1 to 2 scattering)

- LPM regime: many scatterings can cause a gluon to branch with the rate

formation time: $t_f(k) \sim \frac{k}{k_\perp^2} \sim \frac{k}{\hat{q}\, t_f}$

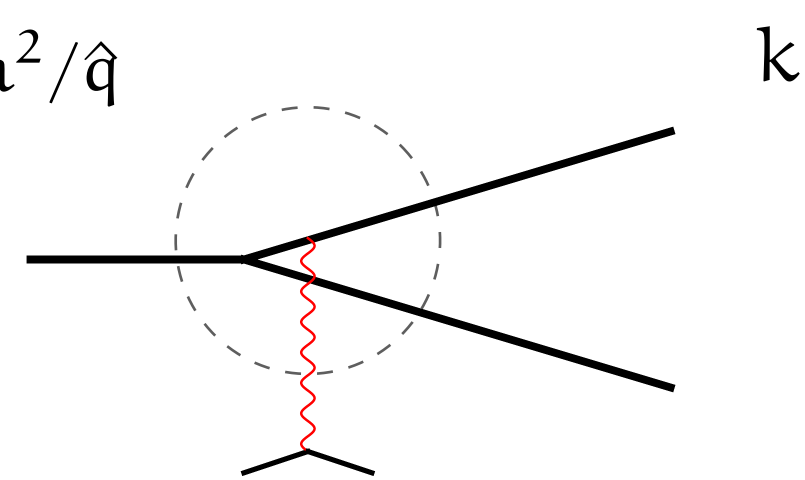
$$k\frac{d\Gamma}{dk} \sim \frac{\alpha}{t_f(k)} \sim \alpha\sqrt{\frac{\hat{q}}{k}}$$



R. Baier, Y. Dokshitzer, A. H. Mueller, S. Peigné, D. Schiff (1995) V. Zakharov (1996)

- Bethe-Heitler regime for $t_f(k) < \ell_{mfp} \sim m^2/\hat{q}$

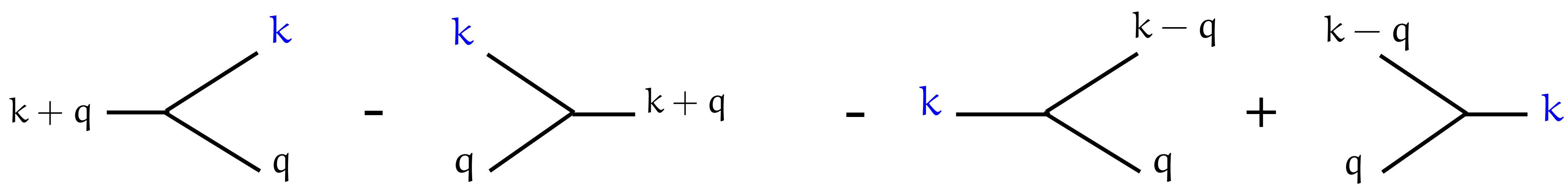
$$k\frac{d\Gamma}{dk} \sim \frac{\alpha}{\ell_{mfp}}$$



J. F. Gunion and G. Bertsch (1982)

Effective 3 waves interaction (1 to 2 scatterings)

$$\frac{\partial}{\partial t} n_k \equiv \frac{1}{k^3}\left[\int_0^{\infty} dq K(k+q,q)F(k+q,q) - \int_0^{k} dq K(k,q)F(k,q)\right]$$



$$F(k,q) \equiv n_{k+q} n_k + (n_{k+q} - n_k) n_q \sim n^2$$

$$K(k,q) \equiv \alpha \sqrt{\hat{q}}\, \frac{(k+q)^{7/2}}{k^{1/2} q^{3/2}}$$

R. Baier, Y. Dokshitzer, A. H. Mueller, D. Schiff, D. T. Son (2000)
P. Arnold, G. D. Moore, L. G. Yaffe (2002)

- Direct energy cascade (if interactions are local!)

$$n_k \sim \frac{P^{1/2}}{\hat{q}^{1/4}\, k^{7/4}}$$

P. Arnold, G. D. Moore (2005)

Non-locality of interactions in momentum space

- Assume a power spectrum $n \sim k^{-x}$ and require the energy flux to be independent of k

- We obtain $x = 7/4$ and

$$P = \alpha \sqrt{\hat{q}} \int_0^1 dz \, \frac{(1-z)^x + z^x - 1}{z^{x+1/2}(1-z)^{x+3/2}} \ln \frac{1}{z}$$

- The above integral diverges at the KZ spectrum

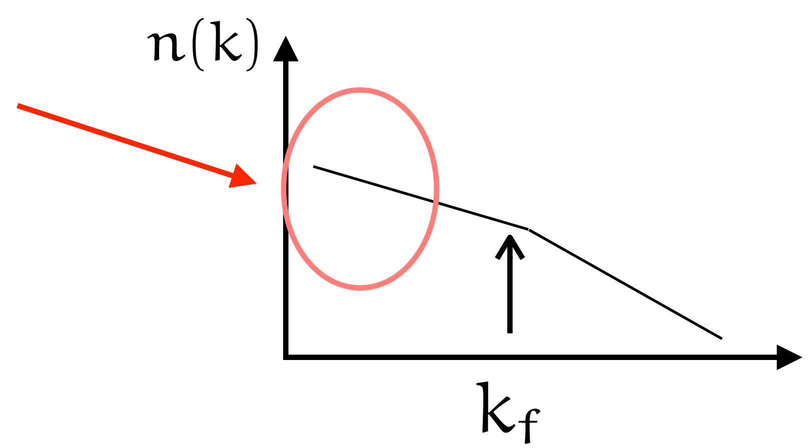
⇒ Effective 3 waves Interaction is nonlocal in momentum space

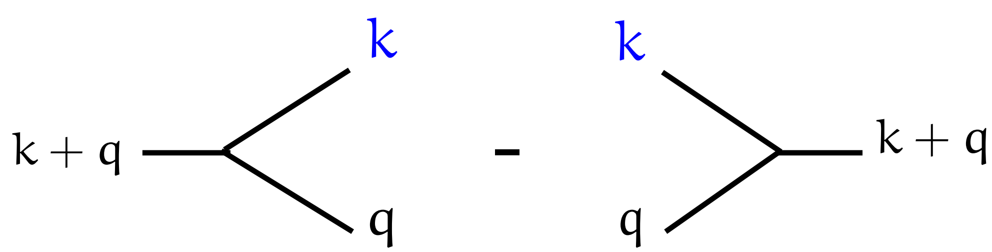
and the KZ spectrum cannot be realized

Thermalization of the soft sector ($k \ll k_f$)

- Nonlocallity ⇒ The collision integral is dominated by strongly asymmetric branchings

- In the regime: $k \ll k_f$ To the left of the source





$$\frac{\partial}{\partial t} n_k \simeq \frac{1}{k^3}\left[\int_0^\infty dq K(k+q,q)F(k+q,q)\right] a \simeq \alpha \frac{\sqrt{\hat{q}}}{k^{7/2}}\left[T_* - k\, n(k)\right]$$

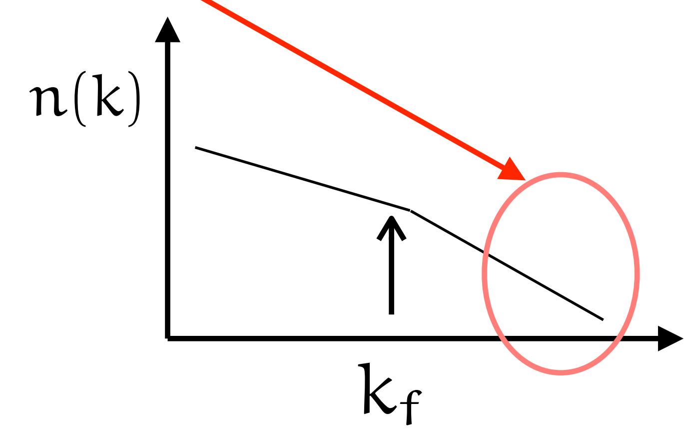
- Late times (steady state) solution is thermal (no fluxes) :

$$n(k) \equiv \frac{T_*}{k}$$

- Therefore, to the left of the forcing the system thermalizes rapidly

Nonlocal energy cascade in the UV ($k \gg k_f$)

- In the regime: $k \gg k_f$ To the right of the source. We perform a gradient expansion around $k \gg q$



- We obtain a diffusion equation in "4-D"

$$\frac{\partial}{\partial t} n(k) \simeq \frac{\hat{q}_{inel}}{4k^3} \frac{\partial}{\partial k} k^3 \frac{\partial}{\partial k} n(k)$$

- with the inelastic diffusion coefficient (in the LPM regime)

$$\hat{q}_{inel} = \alpha \sqrt{\hat{q}} \int_0^\infty dq \sqrt{q}\, n(q)$$

- Straightforward generalization including the BH regime

Nonlocal energy cascade in the UV ($k \gg k_f$)

$$\frac{\partial}{\partial t} n(k) \simeq \frac{\hat{q}_{inel}}{4k^3} \frac{\partial}{\partial k} k^3 \frac{\partial}{\partial k} n(k)$$

- Recall that 3-D diffusion conserves number of particles: $N \sim \int dk\, k^2\, n(k)$

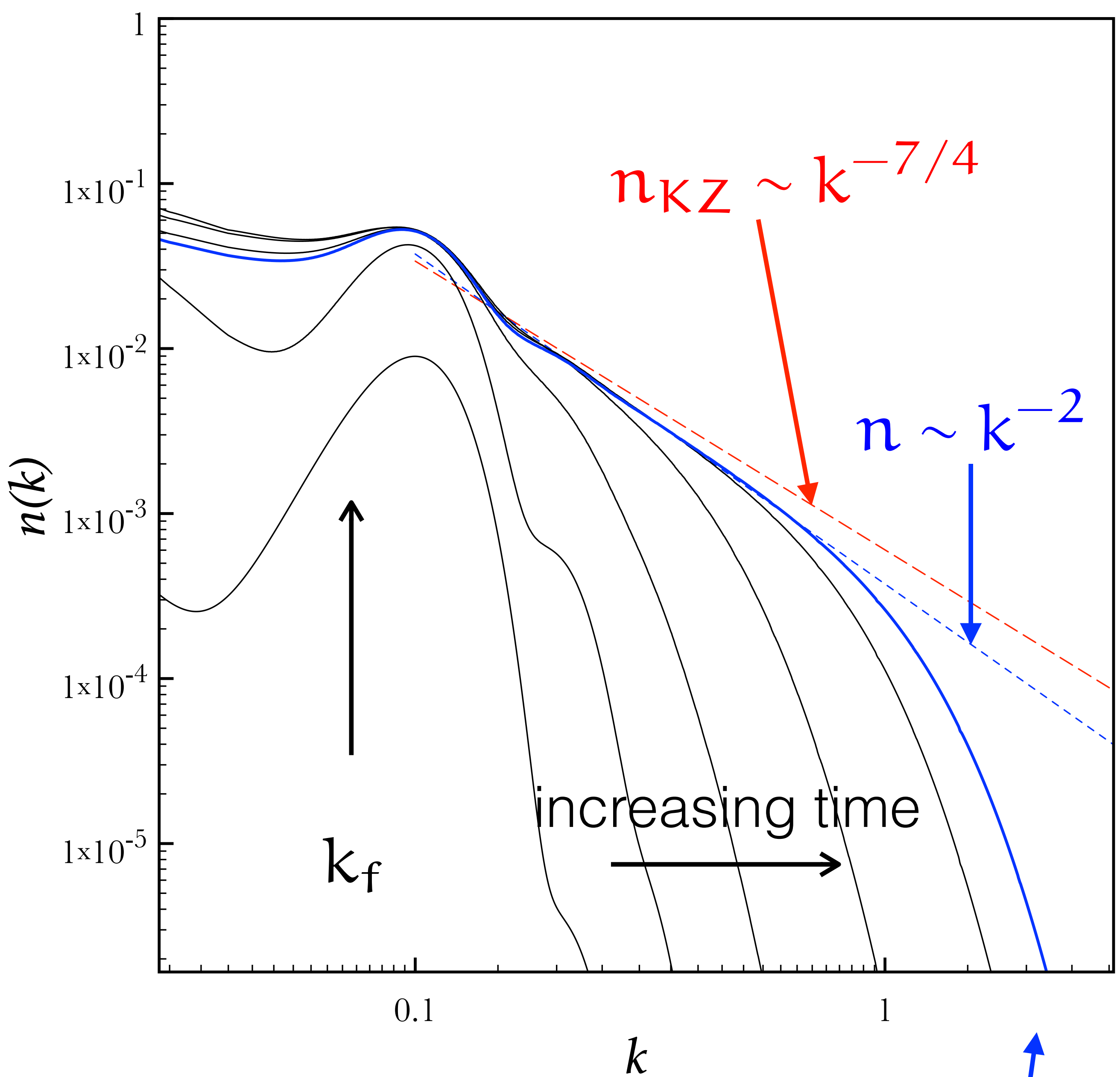
Its fixed point (direct particle cascade): $n(k) \sim \frac{1}{k}$

- 4-D diffusion conserves energy: $E \sim \int dk\, k^3\, n(k)$

Its fixed point (direct energy cascade):

$$n(k) \sim \frac{1}{k^2}$$

Numerical simulation with forcing



wave front evolution: $k_{max}(t) \sim \sqrt{\hat{q}_{inel}\, t}$

- Wave front moving towards the UV leaving in its wake the predicted nonlocal steady state spectrum: hard gluons in the inertial range interact dominantly with gluons at the forcing scale (energy gain)

Parametric estimate:

$$\hat{q} \sim k_f^3 n^2$$

$$\Downarrow$$

$$n(k) \sim \frac{P}{\hat{q} k^2} \sim \frac{P^{1/3} k_f^{1/3}}{k^2}$$

Interplay between elastic and inelastic processes (I)

- For a spectrum falling faster than ***1/k*** one can neglect the drag term in the elastic part. Then, the collision integral in the UV reads

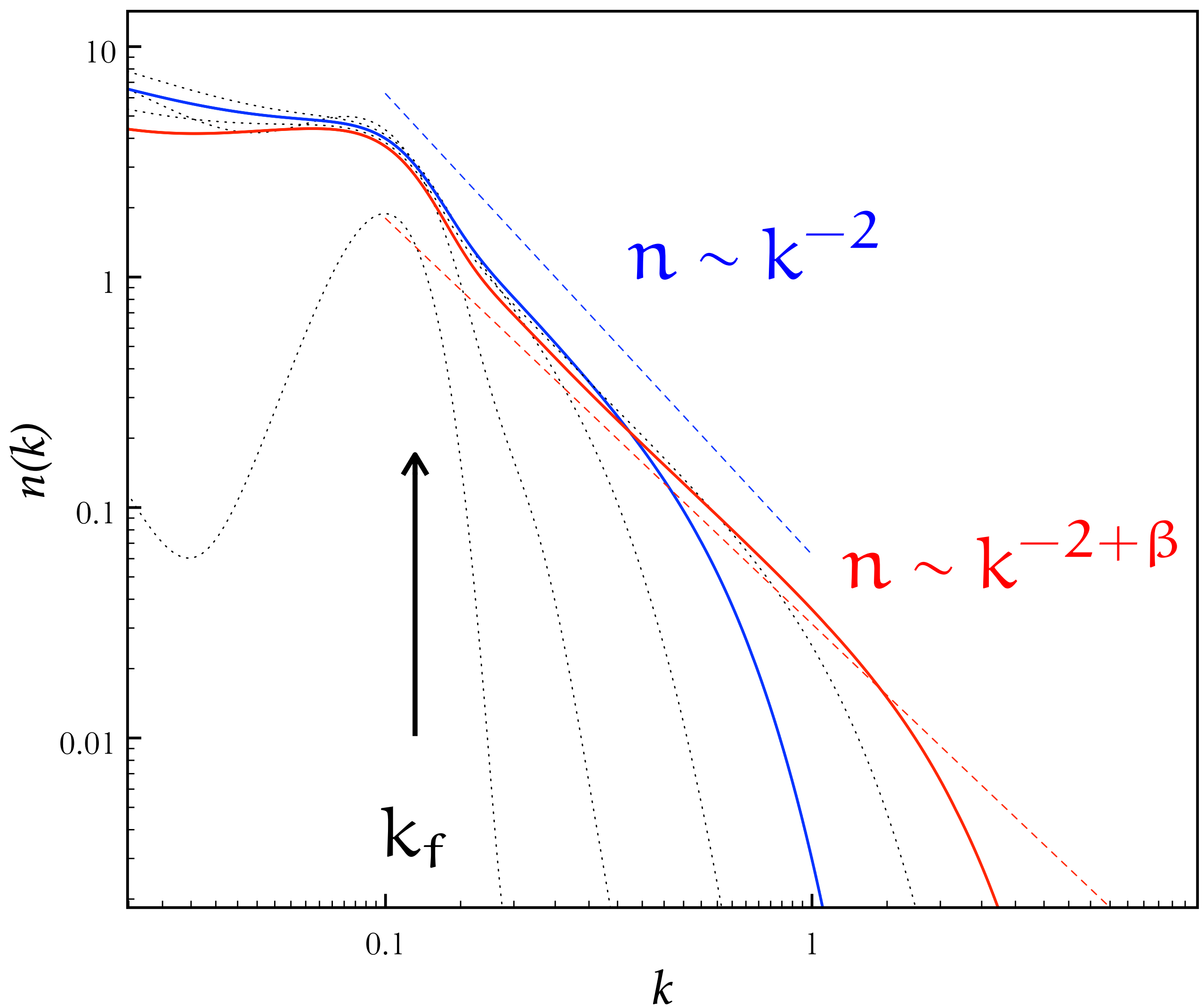
$$\frac{\partial}{\partial t} n(k) \simeq \frac{\hat{q}_{inel}}{4k^3} \frac{\partial}{\partial k} k^3 \frac{\partial}{\partial k} n(k) + \frac{\hat{q}_{el}}{4k^2} \frac{\partial}{\partial k} k^2 \frac{\partial}{\partial k} n(k)$$

$$\equiv \frac{B}{4k^{3-\beta}} \frac{\partial}{\partial k} k^{3-\beta} \frac{\partial}{\partial k} n(k)$$

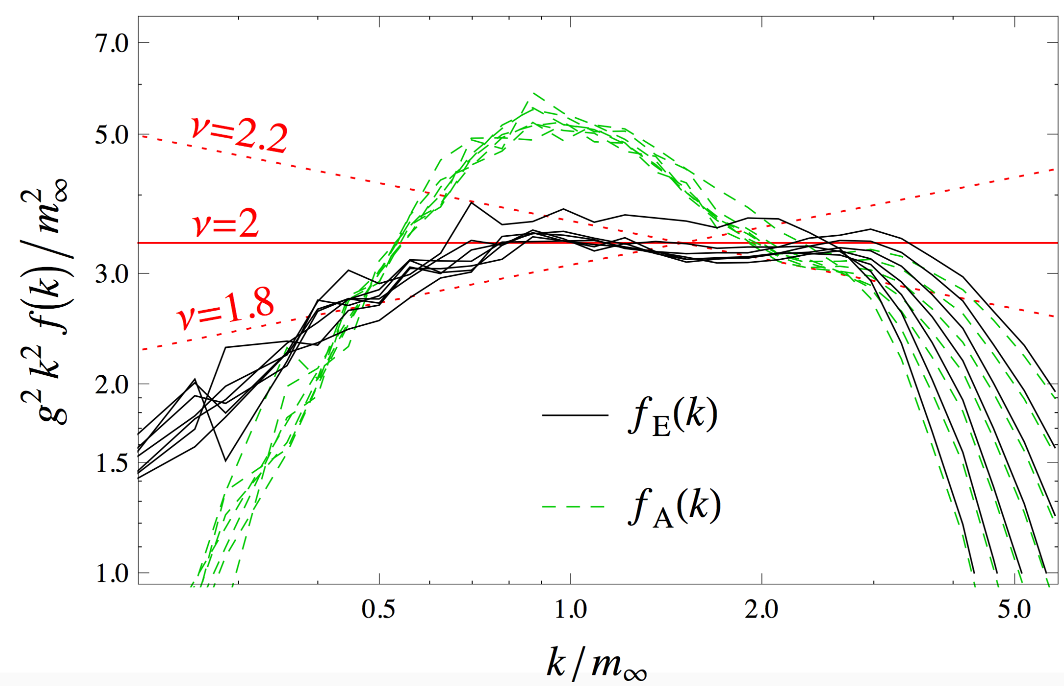
where $B = \hat{q}_{inel} + \hat{q}_{el}$ and $\beta = \dfrac{2}{1 + \hat{q}_{inel}/\hat{q}_{el}}$

- Steady state solution: $n(k) \sim \dfrac{1}{k^{2-\beta}}$ $\quad 0 < \beta < 1$

Interplay between elastic and inelastic processes (II)



A. Ipp, A. Rebhan, M. Strickland (2011)



Exponent $\beta \simeq 0.24$ is computed self-consistently

- Elastic processes reduce slightly the exponent at asymptotically late times. At intermediate times ***k*** $^{-2}$ spectrum is observed: balance between the drag and diffusion terms?

Summary

- Wave turbulence in QCD is different from scalar theories. It is characterized by nonlocal interactions in momentum space: Kolmogorov-Zakharov spectra are not physically relevant
- Inelastic processes dominates the dynamics with a direct energy cascade
- To the left of the forcing scale the system is in thermal equilibrium: warm cascade
- To the right of the forcing scale: kinetic theory predicts a steady state spectrum $\sim k^{-2}$ (in the LPM and BH regimes) in agreement with Hard Loop simulations
- Outlook: mass corrections, anisotropic fluxes, strong turbulence in the presence of strong fields (on the lattice): different exponents?